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# Topological Terms in String Theory on Orbifolds<sup>1</sup>

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## Abstract

We study toroidal orbifold models with topologically invariant terms in the path integral formalism and give physical interpretations of the terms from an operator formalism point of view. We briefly discuss a possibility of a new class of modular invariant orbifold models.

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# 1 Introduction

String theory on toroidal orbifolds [1] has been studied from both operator formalism and path integral formalism points of view. Some of the advantages of the operator formalism are that the spectrum and the algebraic structure are clear and that it is possible to formulate the theory without Lagrangians or actions. On the other hand, in the path integral formalism the geometrical or topological structure is transparent and the generalization to higher genus Riemann surfaces is obvious. Modular invariance of partition functions is rather a trivial symmetry. The interrelation between the two formalisms is not, however, trivial.

In ref.[2], toroidal orbifold models with topologically nontrivial twists have been constructed in the operator formalism. Recently, the construction in the path integral formalism has partly been done in ref.[3]. The main purpose of this paper is to generalize the results of ref.[3] and to construct a wider class of toroidal orbifold models in the path integral formalism by adding new conformally invariant terms to the action.

A  $D$ -dimensional torus  $T^D$  is defined by identifying a point  $\{X^I\}$  with  $\{X^I + \pi w^I\}$  for all  $w^I \in \Lambda$ , where  $\Lambda$  is a  $D$ -dimensional lattice. An orbifold is obtained by dividing the torus by the action of a discrete symmetry group  $P$  of the torus. Any element  $g$  of  $P$  can in general be represented (for symmetric orbifolds) by [1]

$$g = (U, v) , \quad (1.1)$$

where  $U$  denotes a rotation and  $v$  a shift. In the operator formalism of closed string theory, we can introduce a left- and right-moving coordinate  $(X_L^I, X_R^I)$ . On the orbifold, a point  $(X_L^I, X_R^I)$  is identified with  $(U^{IJ}X_L^J + \pi v^I, U^{IJ}X_R^J - \pi v^I)$  for all  $(U, v) \in P$ . If we wish to formulate the orbifold model in the Polyakov path integral formalism [4], we face two problems, as we will explain below. Let  $X^I(\sigma^1, \sigma^2)$  be a string coordinate, which maps a Riemann surface  $\Sigma$  into a target space. In this paper, we will restrict our considerations to a genus one Riemann surface, i.e., a torus. A generalization to higher genus Riemann surfaces will be obvious. The string coordinate on the orbifold in general obeys the following boundary condition:

$$X^I(\sigma^1 + 1, \sigma^2) = U^{IJ}X^J(\sigma^1, \sigma^2) + \pi w^I ,$$

$$X^I(\sigma^1, \sigma^2 + 1) = \tilde{U}^{IJ} X^J(\sigma^1, \sigma^2) + \pi \tilde{w}^I, \quad (1.2)$$

for some  $U, \tilde{U} \in P$  and  $w, \tilde{w} \in \Lambda$ . The consistency of the boundary condition requires

$$[U, \tilde{U}] = 0, \quad (1.3)$$

$$(1 - \tilde{U})^{IJ} w^J = (1 - U)^{IJ} \tilde{w}^J. \quad (1.4)$$

In the Polyakov path integral formalism, the kinetic term is given by

$$S_0[X, g] = \int_0^1 d^2\sigma \frac{1}{2\pi} \sqrt{g(\sigma)} g^{\alpha\beta}(\sigma) \partial_\alpha X^I(\sigma) \partial_\beta X^I(\sigma), \quad (1.5)$$

where  $g_{\alpha\beta}$  is a metric of the Riemann surface  $\Sigma$  of genus one. The kinetic term is conformally invariant and is consistent with the boundary condition (1.2), as it should be. We may add the following topological term to the kinetic one:

$$S_B[X] = i \int_0^1 d^2\sigma \frac{1}{2\pi} B_0^{IJ} \varepsilon^{\alpha\beta} \partial_\alpha X^I(\sigma) \partial_\beta X^J(\sigma), \quad (1.6)$$

where  $\varepsilon^{\alpha\beta}$  is a totally antisymmetric tensor and  $B_0^{IJ}$  is an antisymmetric constant background field, which has been introduced by Narain, Sarmadi and Witten [5] to explain Narain torus compactification [6] in the conventional approach. The first problem is that in the path integral formalism a combination  $X^I = \frac{1}{2}(X_L^I + X_R^I)$  appears in eqs.(1.5) and (1.6) but a combination  $\frac{1}{2}(X_L^I - X_R^I)$  does not. Hence, it seems that there is no way to impose the twisted boundary condition corresponding to the identification  $(X_L^I, X_R^I) \sim (U^{IJ} X_L^J + \pi v^I, U^{IJ} X_R^J - \pi v^I)$  unless  $v^I = 0$  or unless we introduce a new degree of freedom corresponding to  $\frac{1}{2}(X_L^I - X_R^I)$  besides  $X^I$ .

The second problem is concerned with the antisymmetric background field. The integrand of  $S_B[X]$  is not single-valued on the Riemann surface  $\Sigma$  when the twist  $U^{IJ}$  or  $\tilde{U}^{IJ}$  in eq.(1.2) does not commute with  $B_0^{IJ}$  [7, 8]. In ref.[2], orbifold models with such twists have been studied in the operator formalism in detail. The analysis has strongly suggested that those orbifold models belong to a topologically nontrivial class of orbifold models. However, the topological structure has not clearly been understood.

A solution to the above two problems has partly been given in ref.[3]. In this paper, we shall propose a more general solution applicable to a wider class of orbifold models. Our proposal will be given in the next section.

In section 3, we discuss physical meanings of topological terms which we add to the kinetic term, from a path integral formalism point of view. The zero mode part of a one-loop partition function is computed in the path integral formalism.

In section 4, our results in the path integral formalism are reinterpreted from an operator formalism point of view. We see that the interrelation between two formalism is quite nontrivial. Section 5 is devoted to discussion.

## 2 Topological Terms in String Theory on Orbifolds

In this section, we shall propose a solution to the two problems explained in the introduction. A key observation to solve the first problem is a necessity of a new degree of freedom corresponding to  $\frac{1}{2}(X_L^I - X_R^I)$ . Let us introduce a new field variable  $V^I(\sigma)$  which is to be regarded as an external field. Then we have a new conformally invariant term,

$$S_v[X] = i \int_0^1 d^2\sigma \frac{1}{\pi} \varepsilon^{\alpha\beta} \partial_\alpha V^I(\sigma) \partial_\beta X^I(\sigma) . \quad (2.1)$$

Since  $S_v[X]$  is independent of the metric  $g_{\alpha\beta}$ , it is not only conformally invariant but also topologically invariant. For  $S_v[X]$  to be well defined on  $\Sigma$ ,  $V^I(\sigma)$  should obey the following boundary condition:

$$\begin{aligned} V^I(\sigma^1 + 1, \sigma^2) &= U^{IJ} V^J(\sigma^1, \sigma^2) + \pi v^I , \\ V^I(\sigma^1, \sigma^2 + 1) &= \tilde{U}^{IJ} V^J(\sigma^1, \sigma^2) + \pi \tilde{v}^I , \end{aligned} \quad (2.2)$$

for some constant vectors  $v^I$  and  $\tilde{v}^I$ . The consistency of the boundary condition requires

$$(1 - \tilde{U})^{IJ} v^J = (1 - U)^{IJ} \tilde{v}^J . \quad (2.3)$$

Since  $S_v[X]$  is a topological term, it depends only on the boundary conditions (1.2) and (2.2). In terms of zero modes,  $S_v[X]$  can be written as

$$S_v[X] = -i\pi \left( w^I (\tilde{U}^T)^{IJ} \tilde{v}^J - \tilde{w}^I (U^T)^{IJ} v^J \right) . \quad (2.4)$$

We will show in section 4 that a partition function computed in the path integral formalism agrees with that in the operator formalism if the vector  $v^I$  in eq.(2.2) is identified with the shift of the group element  $g = (U, v)$ .

We will next proceed to the second problem. We shall generalize the work of ref.[3]. Let  $\Lambda_R(\mathcal{G})$  ( $\Lambda_W(\mathcal{G})$ ) be a root (weight) lattice of a simply-laced Lie algebra  $\mathcal{G}$  with rank  $D$ . The squared length of the root vectors is normalized to two. In this normalization, the weight lattice  $\Lambda_W(\mathcal{G})$  is just the dual lattice of  $\Lambda_R(\mathcal{G})$ . Instead of  $X^I(\sigma)$ , we may use a new string coordinate  $Z^I(\sigma)$  defined by  $Z^I(\sigma) = M^{IJ}X^J(\sigma)$ , where  $M^{IJ}$  is a constant matrix. Then,  $Z^I(\sigma)$  obeys the following boundary condition:

$$\begin{aligned} Z^I(\sigma^1 + 1, \sigma^2) &= u^{IJ}Z^J(\sigma^1, \sigma^2) + \pi M^{IJ}w^J, \\ Z^I(\sigma^1, \sigma^2 + 1) &= \tilde{u}^{IJ}Z^J(\sigma^1, \sigma^2) + \pi M^{IJ}\tilde{w}^J, \end{aligned} \quad (2.5)$$

where  $u^{IJ} = (MUM^{-1})^{IJ}$  and  $\tilde{u}^{IJ} = (M\tilde{U}M^{-1})^{IJ}$ . It should be noted that  $u^{IJ}$  and  $\tilde{u}^{IJ}$  are not in general orthogonal matrices, although  $U^{IJ}$  and  $\tilde{U}^{IJ}$  are. We may choose  $M^{IJ}$  such that  $M^{IJ}w^J$  (and also  $M^{IJ}\tilde{w}^J$ ) belongs to the lattice  $\Lambda_R(\mathcal{G})$ . This is always possible by appropriately choosing the constant matrix  $M^{IJ}$ . We will restrict our considerations to the case that both  $u^{IJ}$  and  $\tilde{u}^{IJ}$  are orthogonal matrices, otherwise the following discussion will be invalid. We can then apply the results of ref.[3] to our problem. Let us introduce a new field  $\phi(\sigma^1, \sigma^2)$  defined by

$$\phi(\sigma^1, \sigma^2) = \exp\{i2Z^I(\sigma^1, \sigma^2)H^I\}, \quad (2.6)$$

where  $H^I$  is a generator of the Cartan subalgebra of  $\mathcal{G}$  and is normalized such that  $\text{Tr}(H^IH^J) = \delta^{IJ}$ . We note that  $\phi(\sigma^1, \sigma^2)$  is a mapping from  $\Sigma$  into the Cartan subgroup of the group  $G$ , the algebra of which is  $\mathcal{G}$ . A Wess-Zumino term [9]-[12] at level one is given by

$$\Gamma_{WZ}[\tilde{\phi}] = -\frac{i}{12\pi} \int_M \text{Tr}(\tilde{\phi}^{-1}d\tilde{\phi})^3, \quad (2.7)$$

where  $M$  is a three dimensional manifold whose boundary is  $\Sigma$  and  $\phi$  is extended to a mapping  $\tilde{\phi}$  from  $M$  into  $G$  with  $\tilde{\phi}|_{\Sigma} = \phi$ . The Wess-Zumino term is independent of the metric and vanishes for any infinitesimal variation, i.e.,

$$\delta\Gamma_{WZ}[\tilde{\phi}] = -\frac{i}{4\pi} \int_{\Sigma} \text{Tr}(\phi^{-1}\delta\phi(\phi^{-1}d\phi)^2) = 0. \quad (2.8)$$

Thus,  $\Gamma_{WZ}[\tilde{\phi}]$  will depend only on the boundary condition (2.5) or (1.2). We may write the Wess-Zumino term as  $\Gamma_{WZ} = \Gamma_{WZ}(U, w; \tilde{U}, \tilde{w})$ . To this end, we will use the

Polyakov-Wiegmann formula [11],

$$\Gamma_{WZ}[\tilde{\phi}_1\tilde{\phi}_2] = \Gamma_{WZ}[\tilde{\phi}_1] + \Gamma_{WZ}[\tilde{\phi}_2] - \frac{i}{4\pi} \int_{\Sigma} \text{Tr}(\phi_1^{-1} d\phi_1 \phi_2 d\phi_2^{-1}) . \quad (2.9)$$

In terms of the zero modes, the formula (2.9) may be written as

$$\begin{aligned} \Gamma_{WZ}(U, w_1 + w_2 ; \tilde{U}, \tilde{w}_1 + \tilde{w}_2) &= \Gamma_{WZ}(U, w_1; \tilde{U}, \tilde{w}_1) + \Gamma_{WZ}(U, w_2; \tilde{U}, \tilde{w}_2) \\ &- i\pi \left( w_1^I (M^T M U)^{IJ} \tilde{w}_2^J - \tilde{w}_1^I (M^T M \tilde{U})^{IJ} w_2^J \right) \mod 2\pi i. \end{aligned} \quad (2.10)$$

Let us introduce an antisymmetric matrix  $\Delta B^{IJ}$  and a symmetric matrix  $C_U^{IJ}$  through the relations,

$$w^I \Delta B^{IJ} w'^J = w^I (M^T M)^{IJ} w'^J \mod 2 , \quad (2.11)$$

$$w^I C_U^{IJ} w'^J = \frac{1}{2} w^I (\Delta B - U^T \Delta B U)^{IJ} w'^J \mod 2 , \quad (2.12)$$

for all  $w^I, w'^I \in \Lambda$ . It should be noted that the antisymmetric matrix  $\Delta B^{IJ}$  and the symmetric matrix  $C_U^{IJ}$  can always be defined through the relations (2.11) and (2.12) for our choice of the lattice  $\Lambda_R(\mathcal{G})$ . Let us write  $\Gamma_{WZ}$  into the form,

$$\begin{aligned} \Gamma_{WZ}(U, w; \tilde{U}, \tilde{w}) &= i\frac{\pi}{2} w^I C_U^{IJ} w^J + i\frac{\pi}{2} \tilde{w}^I C_U^{IJ} \tilde{w}^J - i\frac{\pi}{2} \tilde{w}^I (U^T \Delta B \tilde{U})^{IJ} w^J \\ &- i\frac{\pi}{2} \tilde{w}^I \Delta B^{IJ} w^J + \Delta\Gamma(U, w; \tilde{U}, \tilde{w}) . \end{aligned} \quad (2.13)$$

Then, it turns out that  $\Delta\Gamma$  would be of the form  $\Delta\Gamma = -i\pi(w^I \tilde{a}^I - \tilde{w}^I a^I)$  modulo  $2\pi i$  for some constant vectors  $a^I$  and  $\tilde{a}^I$ . It follows from eq. (2.4) that  $a^I$  and  $\tilde{a}^I$  can be absorbed into the redefinition of  $v^I$  and  $\tilde{v}^I$ , so that we can assume  $\Delta\Gamma = 0$  without loss of generality. In the following two sections, we will see that the orbifold models with the topological terms  $S_B$  and  $\Gamma_{WZ}$  correspond to those with the antisymmetric background field  $B^{IJ} \equiv B_0^{IJ} + \Delta B^{IJ}$ , which in general does not commute with twists  $U^{IJ}$ .

A simple example discussed just above is the orbifold model with the lattice  $\Lambda = \lambda\Lambda_R(\mathcal{G})$  for some constant  $\lambda$ . The matrix  $M^{IJ}$  may be chosen as  $M^{IJ} = \lambda^{-1}\delta^{IJ}$ . Then, the matrix  $u^{IJ}$  is equal to  $U^{IJ}$  and hence is an orthogonal matrix. Thus, we can apply the above discussion to this orbifold model.

We should make a comment on the Wess-Zumino term. The Wess-Zumino term defined in eq.(2.7) might be modified to make it well defined [13] for some orbifold

models. Our results obtained above crucially rely on the formula (2.9) rather than the expression (2.7) itself. Thus, our results may be valid even if the expression (2.7) is ill defined. What we need is the existence of a term which satisfies the relation (2.9).

### 3 Physical Meanings of the Topological Terms

We have added the three topological terms to the kinetic term. The total action is now given by <sup>1</sup>

$$S[X, g] = S_0[X, g] + S_B[X] + S_v[X] + \Gamma_{WZ}[\tilde{\phi}] . \quad (3.1)$$

In terms of the zero modes, the last three terms in eq.(3.1) can be written as

$$\begin{aligned} S_B + S_v + \Gamma_{WZ} = & i\frac{\pi}{2}w^I C_U^{IJ} w^J + i\frac{\pi}{2}\tilde{w}^I C_U^{IJ} \tilde{w}^J - i\frac{\pi}{2}\tilde{w}^I (U^T B \tilde{U})^{IJ} w^J \\ & - i\frac{\pi}{2}\tilde{w}^I B^{IJ} w^J - i\pi \left( w^I (\tilde{U}^T)^{IJ} \tilde{v}^J - \tilde{w}^I (U^T)^{IJ} v^J \right) , \end{aligned} \quad (3.2)$$

where the antisymmetric matrix  $B^{IJ}$  is defined by

$$B^{IJ} = B_0^{IJ} + \Delta B^{IJ}. \quad (3.3)$$

We note that the definition of  $C_U^{IJ}$  in eq. (2.12) can equivalently be rewritten as

$$w^I C_U^{IJ} w'^J = \frac{1}{2} w^I (B - U^T B U)^{IJ} w'^J \quad \text{mod } 2 , \quad (3.4)$$

for all  $w^I, w'^I \in \Lambda$  since  $B_0^{IJ}$  commutes with  $U^{IJ}$ .

Since the last three terms in the action (3.1) are topological ones, they affect only on zero mode eigenvalues. We will here clarify the effect of the topological terms on zero mode eigenvalues from a path integral formalism point of view. To see this, it may be instructive to recall the Aharonov-Bohm effect in the presence of an infinitely long solenoid [14]. If an electron moves around the solenoid, a wave function of the electron in general acquires a phase. In a path integral formalism point of view, this

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<sup>1</sup> The antisymmetric background field  $B_0^{IJ}$  in  $S_B$  must commute with  $U^{IJ}$  and  $\tilde{U}^{IJ}$ , and the last term  $\Gamma_{WZ}$  cannot always be added to the action, as noticed in the previous sections.

phase is given by the classical action. It may be natural to ask whether Aharonov-Bohm like effects occur in our system. Let us consider a twisted string obeying the boundary condition,

$$X^I(\sigma + 1, \tau) = U^{IJ} X^J(\sigma, \tau) + \pi w^I, \quad (3.5)$$

where  $\tau$  is a “time” coordinate<sup>2</sup>. Suppose that the twisted string moves around the torus, say, from a point  $\{X^I\}$  at  $\tau = 0$  to a point  $\{X^I + \pi \tilde{w}^I\}$  at  $\tau = 1$  for some  $\tilde{w}^I \in \Lambda$ , i.e.,

$$X^I(\sigma, 1) = X^I(\sigma, 0) + \pi \tilde{w}^I. \quad (3.6)$$

The consistency of the boundary conditions (3.5) and (3.6) requires that  $\tilde{w}^I$  must belong to

$$\tilde{w}^I \in \Lambda_U \equiv \{ w^I \in \Lambda \mid w^I = U^{IJ} w^J \} . \quad (3.7)$$

When the twisted string moves around the torus, the wave function  $\Psi(x^I)$  of the string can acquire a phase  $\exp\{-S_B - S_v - \Gamma_{WZ}\}$  in a similar way to the electron moving around the solenoid. Thus we may have<sup>3</sup>

$$\Psi(x^I + \pi \tilde{w}^I) = \rho \Psi(x^I) , \quad (3.8)$$

where

$$\rho = \exp\{-i\pi \tilde{w}^I (-B^{IJ} w^J + v^I + s_U^I)\} . \quad (3.9)$$

The constant vector  $s_U^I$  with  $s_U^I = U^{IJ} s_U^J$  is defined through the relation<sup>4</sup>,

$$\tilde{w}^I s_U^I = \frac{1}{2} \tilde{w}^I C_U^{IJ} \tilde{w}^J \quad \text{mod } 2 , \quad (3.10)$$

for  $\tilde{w}^I \in \Lambda_U$ . To see a physical implication of eq.(3.8), we note that the left hand side of eq.(3.8) may be expressed as

$$\Psi(x^I + \pi \tilde{w}^I) = \exp\{-i\pi \tilde{w}^I \hat{p}_{//}^I\} \Psi(x^I) , \quad (3.11)$$

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<sup>2</sup> Since the topological terms are independent of the world sheet metric, the following arguments do not depend on the signature of the metric. We will here take the Euclidean signature.

<sup>3</sup> We have taken  $\tilde{U} = \mathbf{1}$  and  $\tilde{v} = 0$ .

<sup>4</sup> The existence of such a vector  $s_U^I$  is guaranteed by the fact that  $\frac{1}{2}(w + w')^I C_U^{IJ} (w + w')^J = \frac{1}{2} w^I C_U^{IJ} w^J + \frac{1}{2} w'^I C_U^{IJ} w'^J$  modulo 2 for all  $w^I, w'^I \in \Lambda_U$ .



where  $\hat{p}_{//}^I$  is a canonical momentum operator restricted to the  $U$ -invariant subspace, i.e.,  $\hat{p}_{//}^I = U^{IJ} \hat{p}_{//}^J$ . Comparing eq.(3.8) with eq.(3.11), we conclude that

$$\hat{p}_{//}^I \in 2\Lambda_U^* - B^{IJ} w^J + v^I + s_U^I, \quad (3.12)$$

where  $\Lambda_U^*$  is the dual lattice of  $\Lambda_U$ . In the next section, we will verify the result (3.12) in the operator formalism.

To make the correspondence clear between the path integral formalism and the operator one, we will here give an expression of a one-loop partition function in the path integral formalism. In the next section, we will see that the same partition function can be obtained from the operator formalism. The one-loop partition function  $Z(\tau)$  of the orbifold model will be of the form,

$$Z(\tau) = \frac{1}{|P|} \sum_{\substack{g, h \in P \\ gh = hg}} Z(g, h; \tau), \quad (3.13)$$

where  $\tau$  is the modular parameter and  $|P|$  is the order of the group  $P$ . The  $Z(g, h; \tau)$  denotes the contribution from the string twisted by  $g = (U, v)$  ( $h = (\tilde{U}, \tilde{v})$ ) in the  $\sigma^1$ - ( $\sigma^2$ -) direction. Since  $U\tilde{U} = \tilde{U}U$ , we can take the following coordinate system:

$$\begin{aligned} U^{IJ} &= \begin{pmatrix} \delta^{i_1 j_1} & 0 & 0 & 0 \\ 0 & \delta^{i_2 j_2} & 0 & 0 \\ 0 & 0 & u_3^{i_3 j_3} & 0 \\ 0 & 0 & 0 & u_4^{i_4 j_4} \end{pmatrix}^{IJ}, \\ \tilde{U}^{IJ} &= \begin{pmatrix} \delta^{i_1 j_1} & 0 & 0 & 0 \\ 0 & \tilde{u}_2^{i_2 j_2} & 0 & 0 \\ 0 & 0 & \delta^{i_3 j_3} & 0 \\ 0 & 0 & 0 & \tilde{u}_4^{i_4 j_4} \end{pmatrix}^{IJ}, \\ X^I &= (X^{i_1}, X^{i_2}, X^{i_3}, X^{i_4}), \end{aligned} \quad (3.14)$$

where  $i_k, j_k = 1, 2, \dots, d_k$  ( $k = 1, 2, 3, 4$ ) and  $I, J = 1, 2, \dots, D$  ( $D = d_1 + d_2 + d_3 + d_4$ ). Since the topological terms contribute only to the zero mode part of  $Z(g, h; \tau)$ , it will be enough to give an expression of the zero mode part  $Z(g, h; \tau)_{\text{zero}}$  for our purpose. The result is <sup>5</sup>

$$Z(g, h; \tau)_{\text{zero}} = \int_{\pi\Lambda} dx^I \sum_{w^I \in \Lambda} \sum_{\tilde{w}^I \in \Lambda} \delta_{(1-\tilde{U})^{IJ} w^J, (1-U)^{IJ} \tilde{w}^J} \delta\left(x^{i_2} - \pi\left(\frac{1}{1-\tilde{u}_2}\right)^{i_2 j_2} \tilde{w}^{j_2}\right)$$

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<sup>5</sup>For the details to derive eq.(3.15), see ref.[7].

$$\begin{aligned}
& \times \delta\left(x^{i_3} - \pi\left(\frac{1}{1-u_3}\right)^{i_3 j_3} w^{j_3}\right) \delta\left(x^{i_4} - \pi\left(\frac{1}{1-u_4}\right)^{i_4 j_4} w^{j_4}\right) \\
& \times \exp\left\{-\frac{\pi}{2}\left[\frac{|\tau|^2}{\tau_2}(w^{i_1})^2 + 2\frac{\tau_1}{\tau_2}w^{i_1}\tilde{w}^{i_1} + \frac{1}{\tau_2}(\tilde{w}^{i_1})^2\right]\right. \\
& \quad -i\frac{\pi}{2}w^I C_{\tilde{U}}^{IJ} w^J - i\frac{\pi}{2}\tilde{w}^I C_U^{IJ} \tilde{w}^J + i\frac{\pi}{2}\tilde{w}^I (U^T B \tilde{U})^{IJ} w^J \\
& \quad \left.+ i\frac{\pi}{2}\tilde{w}^I B^{IJ} w^J + i\pi\left(w^I (\tilde{U}^T)^{IJ} \tilde{v}^J - \tilde{w}^I (U^T)^{IJ} v^J\right)\right\}, \quad (3.15)
\end{aligned}$$

where  $\tau = \tau_1 + i\tau_2$ . In the next section, we will see that the phase appearing in eq.(3.15), which comes from the topological terms in the path integral formalism, has a quite different origin in the operator formalism.

## 4 Operator Formalism

In this section, we shall generalize the results of ref.[2] to orbifold models with shifts and clarify physical meanings of the topological terms discussed in the previous sections, from an operator formalism point of view. Although all technical tools have already been developed in ref.[2], it may worth while adding this section since the generalization requires lengthy nontrivial calculations and since the correspondence between the path integral formalism and the operator one is quite nontrivial. For details and notations in this section, see ref.[2].

We first construct the zero mode part of the Hilbert space in the  $g = (U, v)$  twisted sector. It should be emphasized that any topological term does not contribute to the Hamiltonian as well as the equation of motion. In the operator formalism, the antisymmetric background field  $B^{IJ}$  does not appear explicitly in the Hamiltonian but in the following commutation relations of zero modes:

$$\begin{aligned}
[\hat{x}_L^I, \hat{x}_L^J] &= i\pi\left(B - \sum_{m=1}^N \frac{m}{N}(U^{-m} - U^m)\right)^{IJ}, \\
[\hat{x}_R^I, \hat{x}_R^J] &= i\pi\left(B + \sum_{m=1}^N \frac{m}{N}(U^{-m} - U^m)\right)^{IJ}, \\
[\hat{x}_L^I, \hat{x}_R^J] &= i\pi(1 - B)^{IJ}. \quad (4.1)
\end{aligned}$$

where  $N$  is the smallest positive integer such that  $U^N = \mathbf{1}$ . The commutation relations (4.1) are derived from the requirement of the duality of amplitudes. All anomalous

features of the orbifold models originate in eqs.(4.1). We should make a comment on  $B^{IJ}$ . The antisymmetric background field has to satisfy

$$(B - U^T B U)^{IJ} w^J \in 2\Lambda^* \quad \text{for all } w^I \in \Lambda, \quad (4.2)$$

which is the necessary condition for constructing orbifold models in the operator formalism. We note that the condition (4.2) is satisfied for the antisymmetric background field  $B^{IJ} = B_0^{IJ} + \Delta B^{IJ}$  introduced in the path integral formalism. A key to construct the Hilbert space of the zero modes is the following operator:

$$\begin{aligned} I_{k_L, k_R} \equiv & \xi_{k_L, k_R} \exp\{-ik_L^I U^{IJ} \hat{x}_L^J - ik_R^I U^{IJ} \hat{x}_R^J\} \exp\{ik_L^I \hat{x}_L^I + ik_R^I \hat{x}_R^I\} \\ & \times \exp\{i2\pi(k_L^I \hat{p}_{L//}^I - k_R^I \hat{p}_{R//}^I)\}, \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \xi_{k_L, k_R} \equiv & \exp\left\{i\pi\left((k_{L//}^I)^2 - (k_{R//}^I)^2\right) \right. \\ & \left. - i\frac{\pi}{2}(k_L - k_R)^I (UC_U U^T)^{IJ} (k_L - k_R)^J + i\pi(k_L - k_R)^I v^I\right\}. \end{aligned} \quad (4.4)$$

The  $(k_L^I, k_R^I)$  is the left- and right-moving momentum and  $(k_{L//}^I, k_{R//}^I)$  denotes the momentum restricted to the  $U$ -invariant subspace. It is not difficult to show that the operator  $I_{k_L, k_R}$  commutes with all physical operators. Thus,  $I_{k_L, k_R}$  must be a  $c$ -number. We can further show that  $I_{k_L, k_R}$  satisfies the relation  $I_{k_L, k_R} I_{k'_L, k'_R} = I_{k_L + k'_L, k_R + k'_R}$ . Hence,  $I_{k_L, k_R}$  would be of the form  $I_{k_L, k_R} = \exp\{ik_L^I a_L^I - ik_R^I a_R^I\}$  for some constant vectors  $a_L^I$  and  $a_R^I$ . These constant vectors can be determined by requiring the relation  $gV(k_L, k_R; z)g^\dagger = V(k_L, k_R; e^{2\pi i} z)$ , where  $V(k_L, k_R; z)$  is a vertex operator. Then we conclude that  $a_L^I = a_R^I = 0$ , i.e.,

$$I_{k_L, k_R} = 1. \quad (4.5)$$

This identity gives constraints on the eigenvalues of the (restricted) momentum  $\hat{p}_{//}^I$  and the winding number  $\hat{w}^I$ . The identity (4.5) must be satisfied for all momenta  $(k_L^I, k_R^I)$ . It is easy to show that the identity (4.5) for  $k_L^I - k_R^I \in \Lambda_U$  reduces to <sup>6</sup>

$$\exp\left\{i\pi w_{//}^I (\hat{p}_{//}^I + B^{IJ} \hat{w}^J - v^I - s_U^I)_{//} + i\pi k^I \hat{w}^I\right\} = 1, \quad (4.6)$$

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<sup>6</sup>The definition of  $\hat{p}_{//}^I$  in eq.(4.6) is  $\hat{p}_{//}^I = \hat{p}_{L//}^I + \hat{p}_{R//}^I$ , which is different from that in ref.[2] by  $(B^{IJ} \hat{w}^J)_{//}$ .

for all  $w_{//}^I \in \Lambda_U$  and  $k^I \in 2\Lambda^*$ . The identity (4.6) implies that

$$\begin{aligned} (\hat{p}^I + B^{IJ}\hat{w}^J - v^I - s_U^I)_{//} &\in 2\Lambda_U^*, \\ \hat{w}^I &\in \Lambda. \end{aligned} \quad (4.7)$$

This agrees with the result obtained in the previous section. The zero mode eigenstates can be labeled by the eigenvalues of  $(\hat{p}^I + B^{IJ}\hat{w}^J)_{//}$  and  $\hat{w}^I$  as  $|(k^I + v^I + s_U^I)_{//}, w^I\rangle$  for  $k_{//}^I \in 2\Lambda_U^*$  and  $w^I \in \Lambda$ . Due to the existence of the identity operator (4.5), all states  $|(k^I + v^I + s_U^I)_{//}, w^I\rangle$  are not independent. The inner product of the states is quite complicated and is given by

$$\begin{aligned} &\langle (k^I + v^I + s_U^I)_{//}, w^I | (k'^I + v^I + s_U^I)_{//}, w'^I \rangle \\ &= \sum_{\tilde{w}^I \in \Lambda/\Lambda_U} \rho(k, w; \tilde{w}) \delta_{k_{//}^I, (k^I + [U, B]^{IJ}\tilde{w}^J)_{//}} \delta_{w'^I, w^I + (1-U)^{IJ}\tilde{w}^J}, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} \rho(k, w; \tilde{w}) &= \exp \left\{ -i\frac{\pi}{2}\tilde{w}^I C_U^{IJ}\tilde{w}^J + i\pi\tilde{w}_{//}^I (k^I + v^I + s_U^I)_{//} + i\frac{\pi}{2}\tilde{w}^I (U^T B)^{IJ} w_{//}^J \right. \\ &\quad \left. - i\frac{\pi}{2}\tilde{w}_{//}^I B^{IJ} w^J + i\frac{\pi}{2}\tilde{w}^I (B - U^T B U)^{IJ} \left( \frac{1}{1 - \tilde{U}} \right)^{JK}_{\perp} w_{\perp}^K \right. \\ &\quad \left. - i\frac{\pi}{2}\tilde{w}^I (B - U^T B U)^{IJ} \tilde{w}_{//}^J - i\pi\tilde{w}^I (U^T)^{IJ} v^J \right\}. \end{aligned} \quad (4.9)$$

Here,  $w^I$  has been decomposed into two subspaces, i.e.,  $w^I = (w_{//}, w_{\perp})$  and  $\Lambda/\Lambda_U$  in eq.(4.8) denotes the set of the independent lattice points of  $\Lambda$  with the identification  $\tilde{w}^I \sim \tilde{w}^I + \beta^I$  for all  $\beta^I \in \Lambda_U$ .

Another nontrivial feature is an anomalous action of twisted operators on vertex operators. Let  $h = (\tilde{U}, \tilde{v})$  be an element of  $P$ . The action of  $h$  on a vertex operator is found to be

$$hV(k_L, k_R; z)h^{\dagger} = \eta(k_L, k_R; h) e^{i\pi(k_L - k_R)^I \tilde{v}^I} V(\tilde{U}^T k_L, \tilde{U}^T k_R; z), \quad (4.10)$$

where

$$\eta(k_L, k_R; h) = \exp \left\{ -i\frac{\pi}{2}(k_L - k_R)^I (\tilde{U} C_{\tilde{U}} \tilde{U}^T)^{IJ} (k_L - k_R)^J \right\}. \quad (4.11)$$

The anomalous phase  $\eta(k_L, k_R; h)$  is required from the consistency with the commutation relations (4.1). The nontrivial phases in eqs.(4.9) and (4.11) play a crucial role in the following discussion.

We are now ready to compute the one-loop partition function. Let  $Z(g, h; \tau)$  be the partition function of the  $g$ -sector twisted by  $h$  which is defined, in the operator formalism, by

$$Z(g, h; \tau) = \text{Tr} \left[ h e^{i2\pi\tau(L_0 - D/24) - i2\pi\bar{\tau}(\bar{L}_0 - D/24)} \right]_{g\text{-sector}} , \quad (4.12)$$

where  $L_0$  ( $\bar{L}_0$ ) is the Virasoro zero mode operator of the left- (right-) mover. The zero mode part of  $Z(g, h; \tau)$  is computed as <sup>7</sup>

$$\begin{aligned} Z(g, h; \tau)_{\text{zero}} &= \pi^{d_1} \det(1 - \tilde{u}_2) (2\tau_2)^{d_1/2} \\ &\times \sum_{\substack{k_{//}^I \in 2\Lambda_U^* \\ w^I \in \Lambda/(1-U)\Lambda}} \sum_{\tilde{w}^I \in \Lambda/\Lambda_U} \mathcal{A}(g, h; \tau; k_{//}, w, \tilde{w}) , \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \mathcal{A}(g, h; \tau; k_{//}, w, \tilde{w}) &= \delta_{(1-\tilde{U})_{//}^{IJ} (k^J - B^{JK} w^K + v^J + s_U^J)_{//}, 0} \delta_{(1-\tilde{U})^{IJ} w^J, (1-U)^{IJ} \tilde{w}^J} \\ &\times \exp \left\{ i\frac{\pi}{4} \tau (k^I - B^{IJ} w^J + v^I + s_U^I + w^I)_{//}^2 - i\frac{\pi}{4} \bar{\tau} (k^I - B^{IJ} w^J + v^I + s_U^I - w^I)_{//}^2 \right. \\ &\quad - i\frac{\pi}{2} w^I C_{\tilde{U}}^{IJ} w^J - i\frac{\pi}{2} \tilde{w}^I C_U^{IJ} \tilde{w}^J + i\frac{\pi}{2} \tilde{w}^I (U^T B \tilde{U})^{IJ} w^J + i\frac{\pi}{2} \tilde{w}^I B^{IJ} w^J \\ &\quad \left. + i\pi \tilde{w}_{//}^I (k^I - B^{IJ} w^J + v^I + s_U^I)_{//} + i\pi w^I (\tilde{U}^T)^{IJ} \tilde{w}^J - i\pi \tilde{w}^I (U^T)^{IJ} v^J \right\} . \end{aligned} \quad (4.14)$$

It is not difficult to verify that  $Z(g, h; \tau)$  in eq.(4.13) is identical to the expression (3.15) computed in the path integral formalism. All formulas we need to prove the equivalence are given in the appendix of ref.[2].

What we wish to stress is that the equivalence of the partition functions is quite nontrivial because the phases in the partition functions have quite different origins in the path integral formalism and in the operator one: In the path integral formalism, the phase comes from the topological terms which have been added to the kinetic term, while in the operator formalism the phase originates in the nontrivial phases in eqs.(4.8) and (4.10).

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<sup>7</sup>The prefactor in eq.(4.13) has been chosen to agree with  $Z(g, h; \tau)_{\text{zero}}$  given in the path integral formalism.

## 5 Discussion

We have seen that the toroidal orbifold models have topologically quite rich structures. We have studied the orbifold models with the topological terms  $S_B$ ,  $S_v$  and  $\Gamma_{WZ}$ . Adding  $\Gamma_{WZ}$  together with  $S_B$  to the kinetic term is interpreted as the incorporation of the antisymmetric background field  $B^{IJ} = B_0^{IJ} + \Delta B^{IJ}$  in orbifold models, where  $\Delta B^{IJ}$  denotes a topologically nontrivial part of  $B^{IJ}$ . Adding  $S_v$  to the kinetic term is interpreted as the incorporation of the shift  $v^I$  in orbifold models. These interpretations have been verified by comparing the partition functions computed from both the path integral formalism and the operator one.

If we add a new conformally invariant term to the action, we may have a new modular invariant orbifold model, in a path integral formalism point of view. One might add the following term to the action:

$$S_A = i \int_0^1 d^2\sigma \frac{1}{2\pi} A^{IJ} \varepsilon^{\alpha\beta} \partial_\alpha V^I \partial_\beta V^J, \quad (5.1)$$

where  $A^{IJ}$  is an antisymmetric constant matrix satisfying the condition,

$$[A, U] = 0 \quad \text{for all } U \in P, \quad (5.2)$$

and  $V^I$  is the external field introduced in section 2. The term  $S_A$  is conformally, more precisely, topologically invariant and hence does not destroy modular invariance of the partition function. Adding  $S_A$  to the action produces an additional phase factor in the partition function. We have, however, failed to find any orbifold model which produces the same partition function in the operator formalism. We have not known whether the orbifold model with the topological term (5.1) leads to a consistent model from an operator formalism point of view, although it gives a modular invariant partition function in the path integral formalism.

It would be of great interest to look for conformally or topologically invariant terms in string theory compactified on more general manifolds.

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